Recall from last lecture:

$$\left\langle \exp(-S_\Sigma(g)), \bigotimes_{i=1}^{m} \exp(-S_{D_i}(g)) \right\rangle = \exp\left(-S_{\tilde\Sigma}(\tilde g)\right)$$

Let us specify to the case $m=1$ for simplicity.

For a smooth map $f: \Sigma \longrightarrow G_{\mathbb{C}}$ define the "left action" $\ell(f)$ on $\mathcal{L}$ by

$$\ell(f) \exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(f)\right) \bullet \exp\left(-S_\Sigma(g)\right)$$

$$= \exp\left(-S_\Sigma(fg) - T_\Sigma(f,g)\right)$$

Similarly, we define "right action" by

$$r(f) \exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(g)\right) \bullet \exp\left(-S_\Sigma(f)\right)$$

Proposition 2:

Let $g: \Sigma \longrightarrow G_{\mathbb{C}}$ be a smooth map and let $h: \Sigma \longrightarrow G_{\mathbb{C}}$ be a smooth map which is hol. on int. of Σ. Then

$$\ell(h) \exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(hg)\right)$$

and for anti-hol. $h^*: \Sigma \longrightarrow G_{\mathbb{C}}$ we have

$$r(h^*) \exp\left(-S_\Sigma(g)\right) = \exp\left(-S_\Sigma(gh^*)\right)$$

Proof: Use $\bar\partial h = 0$ $\square$

Definition:

A representation $\rho : \mathrm{Map}(\Sigma, G_{\mathbb{C}}) \longrightarrow \mathrm{Aut}(\Gamma(\mathcal{L}))$ (↑ space of sections of $\mathcal{L}$)

is given by

$$[\rho(f)s](\gamma) = \ell(f)\, s((f|_{\partial\Sigma})^{-1}\cdot\gamma),$$

$s \in \Gamma(\mathcal{L})$, $\gamma \in LG_{\mathbb{C}}$, for $f \in \mathrm{Map}(\Sigma, G_{\mathbb{C}})$

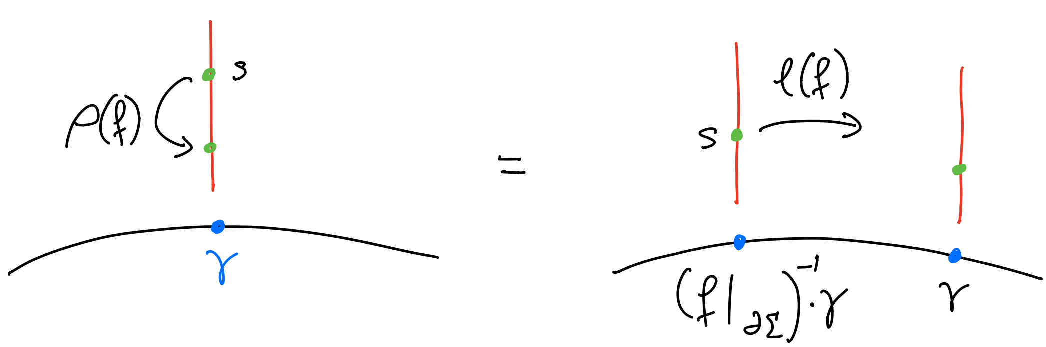
Similarly, define a representation

$\rho^* : \mathrm{Map}(\Sigma, G_{\mathbb{C}}) \longrightarrow \mathrm{Aut}(\Gamma(\mathcal{L}))$ by

$$[\rho^*(f)s](\gamma) = r(f^*)\, s(\gamma\cdot(f^*|_{\partial\Sigma})^{-1}),$$

$s \in \Gamma(\mathcal{L})$, $\gamma \in LG_{\mathbb{C}}$,

where $f^*(z) = {}^t\overline{f(z)}$



Infinitesimal action of $Map(\Sigma, G_{\mathbb{C}})$:

Set for non-negative integer n and $X \in \mathfrak{g}$

$$X_{n,\varepsilon}(z) = e^{\varepsilon X z^n}, \quad z \in D, \quad \varepsilon \in \mathbb{R}$$

and for negative n

$$X_{n,\varepsilon}(z) = e^{\varepsilon X \bar{z}^{-n}}, \quad z \in D, \quad \varepsilon \in \mathbb{R}$$

Infinitesimal action of $X_{n,\varepsilon}$ by ρ is defined by

$$X_n s = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \rho(X_{n,\varepsilon})\, s, \quad s \in \Gamma(\mathcal{L})$$

$\longrightarrow$ The map defined by $X \otimes t^n \mapsto X_n$ gives representation of affine Lie algebra $\hat{\mathfrak{g}}$ on space of sections $\Gamma(\mathcal{L})$:

Lemma:

The operators X_m and Y_n, $m, n \in \mathbb{Z}$, satisfy the relation

$$[X_m, Y_n] = [X, Y]_{m+n} + mk\, \delta_{m+n,0} \langle X, Y \rangle$$

Proof:

Put $f = X_{m,\varepsilon_1}$ and $g = Y_{n,\varepsilon_2}$ for $\varepsilon_1, \varepsilon_2 \in \mathbb{R}$

In the case $m,n \geq 0$ or $m,n \leq 0$ the relation

$$[X_m, Y_n] = [X, Y]_{m+n}$$

follows from

$$T_D(f,g) = T_D(g,f) = 0.$$

Let us suppose $m \geq 0$ and $n \leq 0$. Then

$$T_D(f,g) = 0, \quad \text{but}$$

$$\lim_{\varepsilon_1, \varepsilon_2 \to 0} \frac{1}{\varepsilon_1 \varepsilon_2} T_D(g,f) = \frac{K}{2\pi\sqrt{-1}} \int_D \mathrm{Tr}\left(m z^{m-1} X dz \wedge n \bar{z}^{-n-1} Y d\bar{z} \right)$$

$$= mK \langle X, Y \rangle$$

for $m = -n$ and zero otherwise. $\square$

Define operators $\overline{X}_n s = \frac{d}{d\varepsilon}\Big|_{\varepsilon=0} \rho^*(X_{n,\varepsilon}) s, \; s \in \Gamma(\mathcal{L})$

Then $X \otimes t^n \mapsto \overline{X}_n$ also defines a repr. of $\hat{\mathfrak{g}}$

$$\longrightarrow [\overline{X}_m, \overline{Y}_n] = \overline{[X, Y]}_{m+n} + mK \delta_{m+n,0} \langle X, Y \rangle$$

<u>but:</u> $[X_m, \overline{Y}_n] = 0$

Definition:

A smooth section $\varphi \in \Gamma(\mathcal{L})$ is called "primary" if and only if

$$X_n \varphi = \overline{X_n} \varphi = 0 \quad \forall\, X \in \mathfrak{g},\ n > 0$$

One can show that the space of sections of $\mathcal{L}$ contains a subspace in the representation $\bigoplus_{0 \le \lambda \le k} H_\lambda \otimes H_\lambda^*$

Physics interpretation:

For a closed Riemann surface $\widetilde{\Sigma}$ consider

$$\int_{f : \widetilde{\Sigma} \to G_{\mathbb{C}}} \exp(-S_{\widetilde{\Sigma}}(f)) \, \mathcal{D}f$$

Decompose $\widetilde{\Sigma} = \Sigma \cup \bigoplus_{i=1}^{m} D_i$. Take $m=1$.

$$\exp(-S_\Sigma(f)) \in \pi^{-1}(f \circ \iota) \quad \text{where } \pi : \mathcal{L} \to LG_{\mathbb{C}}$$

This section will be an element of $\bigoplus_{0 \le \lambda \le k} H_\lambda \otimes H_\lambda^*$

Then the path integral over $\widetilde{\Sigma}$ will be obtained through the pairing $\mathcal{L} \times \mathcal{L}^{-1} \to \mathbb{C}$

§4. The space of conformal blocks and fusion rules

Consider the Riemann sphere $\mathbb{CP}^1$ with homogeneous coordinates $[\zeta_0 : \zeta_1] \longrightarrow z = \zeta_0/\zeta_1$

identify $\mathbb{CP}^1 \cong \mathbb{C} \cup \{\infty\}$

Let $p_1, \dots, p_n \in \mathbb{CP}^1$ be n distinct points

introduce coordinates $z_j = z(p_j)$ at p_j.

$\longrightarrow$ locally $t_j = z - z_j$ for $p_j \neq \infty$

for $p_j = \infty$ take $t_j = \frac{1}{z}$

Now suppose $p_j \neq \infty \; \forall j$

Denote by $\mathcal{M}_{p_1, \dots, p_n}$ the vector space of meromorphic functions on $\mathbb{CP}^1$ with poles of any order at most at $p_1, \dots, p_n$. Set

$$\mathfrak{g}(p_1, \dots, p_n) = \mathfrak{g} \otimes \mathcal{M}_{p_1, \dots, p_n}$$

where $\mathfrak{g}$ is $\mathfrak{sl}(2, \mathbb{C})$ Lie algebra.

$\longrightarrow$ $\mathfrak{g}(p_1, \dots, p_n)$ has structure of Lie algebra:

$$[X \otimes f, Y \otimes g] = [X, Y] \otimes fg, \quad X, Y \in \mathfrak{g}; \; f, g \in \mathcal{M}_{p_1, \dots, p_n}$$

$\longrightarrow$ Laurent expansion of elements of $\mathfrak{g}(p_1, \dots, p_n)$ at p_j with respect to t_j gives linear map

$$\tau_j : \mathfrak{g}(p_1, \dots, p_n) \longrightarrow \mathfrak{g} \otimes \mathbb{C}((t_j))$$

for each j, $1 \le j \le n$. Injection of $\mathfrak{g} \otimes \mathbb{C}((t_j))$ to affine Lie algebra $\hat{\mathfrak{g}}_j = \mathfrak{g} \otimes \mathbb{C}((t_j)) \otimes \mathbb{C}$ then gives:

$$\iota_j : \mathfrak{g}(p_1, \dots, p_n) \longrightarrow \hat{\mathfrak{g}}_j .$$

Fix level κ. Associate $p_j \longrightarrow H_{\lambda_j}$ (integrable highest weight module)

<u>Definition</u>:

The diagonal action Δ of $\mathfrak{g}(p_1, \dots, p_n)$ on $H_{\lambda_1} \otimes \cdots \otimes H_{\lambda_n}$ is given by

$$\Delta(\varphi)(\xi_1 \otimes \cdots \otimes \xi_n) = \sum_{j=1}^{n} \xi_1 \otimes \cdots \otimes \iota_j(\varphi)\xi_j \otimes \cdots \otimes \xi_n$$

for $\varphi \in \mathfrak{g}(p_1, \dots, p_n)$ and $\xi_j \in H_{\lambda_j}$, $1 \le j \le n$.

Lemma:

The above action
$$\Delta : \mathfrak{g}(p_1, \dots, p_n) \longrightarrow \mathrm{End}(H_{\lambda_1} \otimes \dots \otimes H_{\lambda_n})$$
is representation of the Lie algebra $\mathfrak{g}(p_1, \dots, p_n)$

Proof:

For $f \in \mathcal{M}_{p_1, \dots, p_n}$ denote by f_{p_j} the Laurent series in t_j at p_j. The 2-cocycle ω introduced in §1, Proposition 1, satisfies:
$$\sum_{j=1}^{n} \omega(X \otimes f_{p_j}, Y \otimes g_{p_j}) = 0$$
for any $X \otimes f, Y \otimes \mathfrak{g}(p_1, \dots, p_n)$ since
$$\sum_{j=1}^{n} \omega(X \otimes f_{p_j}, Y \otimes g_{p_j}) = \langle X, Y \rangle \sum_{j=1}^{n} \mathrm{Res}_{t_j=0}(df g)$$
and sum of residues of a meromorphic 1-form is zero.
$$\longrightarrow \Delta([X, Y] \otimes fg) = [\Delta(X \otimes f), \Delta(Y \otimes g)]$$
$\square$